hep-th/9606187

Quantum corrections of Abelian Duality Transformations

J. Balog*
MPI, München, P.O. Box 40 12 12, München, Germany
P. Forgács†
Laboratoire de Math. et Physique Théorique, Université de Tours,
Parc de Grandmont, F-37200 Tours, France
Z. Horváth‡and L.Palla§
Institute for Theoretical Physics, Roland Eötvös University,
H-1088, Budapest, Puskin u. 5-7, Hungary

February 7, 2008

Abstract

A modification of the Abelian Duality transformations is proposed guaranteeing that a (not necessarily conformally invariant) σ-model be quantum equivalent (at least up to two loops in perturbation theory) to its dual. This requires a somewhat non standard perturbative treatment of the *dual* σ-model. Explicit formulae of the modified duality transformation are presented for a special class of block diagonal purely metric σ-models.

PACS codes: 02.40-k, 03.50.Kk, 03.70, 11.10.L, 11.10.Kk
key words: sigma models, duality, quantum corrections

*E-mail: balog@mppmu.mpg.de
†E-mail: forgacs@celfi.phys.univ-tours.fr
‡E-mail: zalanh@ludens.elte.hu
§E-mail: palla@ludens.elte.hu

1 Introduction

Various T ('target space') duality transformations [1], [2], [3] connecting two seemingly different σ-models or string-backgrounds have aroused a considerable amount of interest. In Ref. [4] we have investigated quite a few examples of dually related 'ordinary' (i.e. not necessary conformally invariant) σ-models treated as two dimensional quantum field theories in the framework of perturbation theory. We have shown on a number of examples that the 'naive' (tree level) T-duality transformations [1] *cannot be* exact symmetries of the quantum theory. The 'naive' Abelian duality transformations yield a model equivalent to the original one only to one loop order in perturbation theory, however, the equivalence breaks down in general, at the two loop order. We reached these conclusions by comparing various β functions in the original and dual theories. Therefore it seems to be clear that the question of quantum equivalence between dual σ-models deserves further study.

The quantum equivalence of dually related Conformal Field Theories (CFT) has been proven in Ref. [5]. For a number of special cases (Wess-Zumino-Witten (WZW) models [6], gauged WZW models [7] and two dimensional black holes [8]) this equivalence has been shown explicitly.

To fix ideas let us first write the two dimensional σ-model action:

$$S=\frac{1}{4\pi\alpha'}\int d^2z\Big[\sqrt{h}h^{\mu\nu}\left(g_{00}\partial_\mu\theta\partial_\nu\theta+2g_{0\alpha}\partial_\mu\theta\partial_\nu\xi^\alpha+g_{\alpha\beta}\partial_\mu\xi^\alpha\partial_\nu\xi^\beta\right) \\ +i\epsilon^{\mu\nu}(2b_{0\alpha}\partial_\mu\theta\partial_\nu\xi^\alpha+b_{\alpha\beta}\partial_\mu\xi^\alpha\partial_\nu\xi^\beta)\Big] \tag{1}$$

where g_{ij} is the target space metric, b_{ij} is the (antisymmetric) torsion potential, $h_{\mu\nu}$ is the world sheet metric and α' the inverse of the string tension. In Eq. (1) we have assumed that there is a Killing vector and in the adopted coordinate system, the target space indices are decomposed as $i=(0,\alpha)$ corresponding to splitting the coordinates as $\xi^i=(\theta,\xi^\alpha)$, and then the background fields $(g\,,b)$ are independent of the coordinate $\xi^0=\theta$. Note the absence of the dilaton field in Eq. (1). In this letter we concentrate mainly on not conformally invariant σ-models, quantized as ordinary quantum field theories. In the same spirit the world sheet metric, $h_{\mu\nu}$, is taken to be flat in what follows.

Now the well known formulae of Abelian T-duality [1], mapping the 'original' σ-model with action, $S[g\,,b]$, given in Eq. (1) to its dual, $S[\tilde{g}\,,\tilde{b}]$ are:

$$\tilde{g}_{00}=\frac{1}{g_{00}}\qquad \tilde{g}_{0\alpha}=\frac{b_{0\alpha}}{g_{00}}\,,\qquad \tilde{b}_{0\alpha}=\frac{g_{0\alpha}}{g_{00}}\,, \\ \tilde{g}_{\alpha\beta}=g_{\alpha\beta}-\frac{g_{0\alpha}g_{0\beta}-b_{0\alpha}b_{0\beta}}{g_{00}}\,,\quad \tilde{b}_{\alpha\beta}=b_{\alpha\beta}-\frac{g_{0\alpha}b_{0\beta}-g_{0\beta}b_{0\alpha}}{g_{00}}\,. \tag{2}$$

It has been recently found that the Abelian duality transformation rules (2) can be recovered in an elegant way – without ever using the dilaton – by performing a canonical transformation [9]. This clearly shows that the models related by these transformations are *classically* equivalent. In the quantum theory, the usual way to argue that the dually related models are equivalent in spite of the non linear change of variables involved, is by making some formal manipulations

in the functional integral [1], ignoring the need for regularization. For a special class of *conformally invariant* σ-models (string backgrounds) it has already been found in Ref. [10] that the Abelian T-duality transformations rules of Ref. [1] should be modified at the two loop level to preserve conformal invariance.

The aim of this letter is to put forward a nontrivial modification of the standard Abelian T-duality transformations, Eqs. (2), which should promote them to a full *quantum* symmetry. The basic motivation for such a modification is easy to understand; the *bare* and the *renormalized* quantities do not transform in the same way under duality transformations beyond one loop order in perturbation theory. While our proposed modification of the T-duality transformation rules is certainly necessary to ensure that this symmetry hold in the quantum theory, it implies that the 'naive' duality transformations receive perturbative corrections order by order (beyond one loop). Even more interestingly the modified duality transformations do not map σ-models into σ-models in the usual sense, except for the class of *conformally invariant* models (or string backgrounds).

We illustrate how the proposed modifications ensure two loop equivalence between the original and its dual on an example of an asymptotically free σ-model (the O(3) model) and on the example of two free fields, written in polar coordinates, both cases treated as ordinary quantum field theories.

2 Modified duality transformations

When deriving the Abelian duality transformations all formal manipulations are carried out on *unrenormalized*, i.e. *bare* quantities. The partition function of a generic σ-model using dimensional regularization can be written as:

$$Z = \int D\xi^i \exp\Big(-\frac{\mu^{-\epsilon}}{4\pi\alpha'}\int d^{2-\epsilon}z\, T^{(0)}_{ij}(g,b)\Xi^{ij}\Big)\,, \tag{3}$$

where $\Xi^{ij} = (\partial_\mu\xi^i\partial^\mu\xi^j + i\epsilon_{\mu\nu}\partial^\mu\xi^i\partial^\nu\xi^j)$, and the generalized bare metric, $T^{(0)}_{ij} = g^{(0)}_{ij} + b^{(0)}_{ij}$, has been computed in terms of the renormalized quantities (g_{ij}, b_{ij}) by several authors [11], [12], [13], by the background field method in the dimensional regularization scheme:

$$T^{(0)}_{ij}(g,b) = g_{ij} + b_{ij} + \frac{\alpha'}{\epsilon}\hat{R}_{ij}(g,b) + \frac{(\alpha')^2}{\epsilon}\hat{Y}_{ij}(g,b) + \ldots\,, \tag{4}$$

where

$$\begin{aligned} \hat{Y}_{ij} =& \frac{1}{8}Y^{lmk}{}_j\hat{R}_{iklm}\,, \\ Y_{lmkj} =& -2\hat{R}_{lmkj} + 3\hat{R}_{[klm]j} + 2(H^2_{kl}g_{mj} - H^2_{km}g_{lj})\,, \\ H^2_{ij} =& H_{ikl}H^{kl}_j\,, \qquad 2H_{ijk} = \partial_i b_{jk} + \text{cyclic}\,. \end{aligned} \tag{5}$$

In Eqs. (4, 5) $\hat{R}_{ij}$ resp. $\hat{R}_{iklm}$ denote the 'generalized' Ricci resp. Riemann tensors of the 'generalized' connection, G^i_{jk}, containing also the torsion term in addition to the Christoffel symbols, Γ^i_{jk} of the metric g_{ij}; $G^i_{jk} = \Gamma^i_{jk} + H^i_{jk}\,.$ A very natural idea would be to perform the 'naive' duality transformations

(2) on the bare quantities, $T^{(0)}_{ij}(g,b)$, that is to impose as the quantum duality symmetry:

$$\tilde{T}^{(0)}_{ij}(g,b) = T^{(0)}_{ij}(\tilde{g},\tilde{b})\,, \tag{6}$$

where $\tilde{}$ denotes the transformation defined by Eq. (2), (the symmetric part of $T^{(0)}_{ij}$, $T^{(0)}_{(ij)}$, transforms as the metric, g_{ij}, while the antisymmetric part, $T^{(0)}_{[ij]}$, as b_{ij}). For example

$$\tilde{T}^{(0)}_{00}(g,b) = \tilde{g}^{(0)}_{00} = \frac{1}{g_{00} + \alpha'/\epsilon \hat{R}_{00}(g,b) + \ldots} = T^{(0)}_{00}(\tilde{g},\tilde{b})\,. \tag{7}$$

In fact Eqs. (6) should only hold modulo diffeomorphisms (redefinitions of the target space coordinates).

For all the examples studied in [4] it has been found that in the one loop order the original and the dual models are equivalent after the field redefinition (reparametrization): $\xi^i_0 \to \xi^i + \alpha' \xi^i_1(\xi)/\epsilon$, $\xi^i_1(\xi) \sim \partial_i \ln g_{00}(\xi)$; (see Eq. (24) below), implying that (at least for the cases in question) Eqs. (6) hold. Comparing the coefficients of α'/ϵ on the two sides of (6) one finds that the generalized Ricci tensors computed from the original and dual quantities should be related – up to a reparametrization – as:

$$\begin{aligned}
\hat{R}^{\tilde{g}}_{00} &= -\frac{1}{g^2_{00}}\hat{R}^{g}_{00}, \quad \hat{R}^{\tilde{g}}_{(0\alpha)} = -\frac{1}{g^2_{00}}\left(b_{0\alpha}\hat{R}^{g}_{00} - \hat{R}^{b}_{[0\alpha]}g_{00}\right)\,,\\
\hat{R}^{\tilde{b}}_{[0\alpha]} &= -\frac{1}{g^2_{00}}\left(g_{0\alpha}\hat{R}^{g}_{00} - \hat{R}^{g}_{(0\alpha)}g_{00}\right)\,,\\
\hat{R}^{\tilde{g}}_{(\alpha\beta)} &= \hat{R}^{g}_{(\alpha\beta)} - \frac{1}{g_{00}}\left(\hat{R}^{g}_{(0\alpha)}g_{0\beta} + \hat{R}^{g}_{(0\beta)}g_{0\alpha} - \hat{R}^{b}_{[0\alpha]}b_{0\beta} - \hat{R}^{b}_{[0\beta]}b_{0\alpha}\right)\\
&\quad + \frac{1}{g^2_{00}}\left(g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta}\right)\hat{R}^{g}_{00}\,,\\
\hat{R}^{\tilde{b}}_{[\alpha\beta]} &= \hat{R}^{b}_{[\alpha\beta]} - \frac{1}{g_{00}}\left(\hat{R}^{g}_{(0\alpha)}b_{0\beta} + \hat{R}^{b}_{[0\beta]}g_{0\alpha} - \hat{R}^{g}_{(0\beta)}b_{0\alpha} - \hat{R}^{b}_{[0\alpha]}g_{0\beta}\right)\\
&\quad + \frac{1}{g^2_{00}}\left(g_{0\alpha}b_{0\beta} - b_{0\alpha}g_{0\beta}\right)\hat{R}^{g}_{00}\,.
\end{aligned} \tag{8}$$

Quite recently a general proof of the validity of Eqs. (8) has been given in Ref. [14], implying that up to the one loop order Abelian T-duality as defined by Eqs. (6) holds, indeed. Quantum equivalence under duality transformation means that the functional integrals computed with either $T^{(0)}_{ij}(g,b)$ or with $\tilde{T}^{(0)}_{ij}(g,b)$ should lead to identical results *to any desired order* in perturbation theory (for physical quantities of course). The examples studied in Ref. [4] show, however, that in general the functional integrals computed with $T^{(0)}_{ij}(g,b)$ and $T^{(0)}_{ij}(\tilde{g},\tilde{b})$ lead to different physics beyond one loop. This also implies that Eqs. (2.4) of Ref. [14] (which are equivalent to the 'naive' duality equations (6)) will not be consistent for a general background $(g\,,b)$ at the two loop level. One might ask, how this (somewhat discouraging) result complies with the results of Ref. [5] showing that duality maps a conformal field theory (string background) into a string background. First the examples of Ref. [4] do not correspond to

string backgrounds, but even more importantly the proof of Ref. [5] is based on a gauging proceedure of chiral currents, and there is no claim whatsoever that the 'naive' transformation formulae Eqs. (2) would be exact to all orders in α'. In fact as already mentioned, in Ref. [10] for certain string backgrounds it has been explicitly shown that the Abelian T-duality transformation rules, Eqs. (2), have to be modified at the two loop level.

Let us now present our modified transformation rules for the Abelian T-duality transformations, which should make them a true quantum symmetry, valid to all orders in perturbation theory. Instead of Eqs. (6) we postulate the following equation for a *finite* mapping, $\gamma(g, b)$:

$$\tilde{T}^{(0)}(g,b) = \left(\gamma^{-1} \circ T^{(0)} \circ \gamma\right)(\tilde{g}, \tilde{b})\,, \tag{9}$$

where equality is meant again *modulo reparametrizations* of the target space coordinates. If such a γ exists for any σ-model background (g, b) then we would say that the classical duality symmetry is a true symmetry of the full quantum theory. The modified dual (or quantum dual) of a σ-model is defined as

$$(g,b)_q = \gamma(\tilde{g}, \tilde{b})\,. \tag{10}$$

Eq. (9) expresses the way in which the renormalization and the renormalized metric and torsion potential change under a transformation of the bare quantities. In this respect Eq. (9) is in complete analogy with the equation that describes the change in the renormalization of an ordinary *parameter*, whose bare and renormalized values are related as $e_0 = Z(e)$. When one changes from e_0 and e to $\hat{e}_0 \equiv f(e_0)$ and $\hat{e} \equiv f(e)$ the relation between the new bare and renormalized parameters becomes $\hat{e}_0 = f(Z(f^{-1}(\hat{e})))$. One might be tempted to interpret Eq. (9) as the action of the duality transformation on the *renormalized metric* combined with a change of the renormalization scheme [15]. It is well known that the β-functions in general are scheme dependent beyond one loop. In our case, however, Eq. (9) is only defined for backgrounds possessing an Abelian isometry, and therefore it is not obvious if the above interpretation is correct. In fact as shown for a special class of metrics, Eq. (9) does not correspond to a change of scheme compatible with *full* target space covariance.

At this point we note that while it is very natural to assume that the existence of a non-trivial mapping, γ, would guarantee that duality is indeed a quantum symmetry, the dual model defined by either sides of Eq. (9) *does not correspond* any longer to a genuine σ-model as the standard relation between the bare and the renormalized metric and torsion given by Eqs. (4) is lost. A simple consequence of Eq. (9) for the mapping γ is:

$$\widetilde{\gamma^{-1}}(g) = \gamma(\tilde{g})\,. \tag{11}$$

¿From Eq. (9) it also immediately follows that the modified duality transformation maps conformal σ-models into conformal σ-models in contrast to the general case. For conformal σ-models the β-functions vanish, therefore the metric and torsion, (g, b), in this case, satisfies

$$T^{(0)}(g,b) = (g,b)\,, \tag{12}$$

where again Eq. (12) is supposed to hold only modulo a diffeomorphism. It is now easy to see that the quantum dual of a conformal σ-model is again a conformal σ-model, indeed. By acting with γ on both sides of Eq. (9) one obtains:

$$\left(\gamma \circ \tilde{T}^{(0)}\right)(g,b) = T^{(0)}\left((g,b)_q\right) , \tag{13}$$

showing that the modified duality transformation maps conformally invariant models onto themselves.

At present we can only analyse Eqs. (9) in the general case in perturbation theory. This way one determines $\gamma(g,b)$ order by order in the α' expansion, that is

$$\gamma_{ij}(g,b) = g_{ij} + b_{ij} + \alpha' M_{ij}(g,b) + \dots . \tag{14}$$

Then Eqs. (9) will connect order by order in α' $\tilde{T}^{(0)}_{ij}(g,b)$ and $T^{(0)}(\tilde{g},\tilde{b})$ [1]. Therefore if M_{ij} turns out to be non trivial it means that the naive duality transformations, Eq. (2), must be modified in higher orders. Note, that M_{ij}, the first nontrivial terms of γ_{ij}, have no effect on the one loop results: indeed using Eqs. (2), (9) we find from comparing the coefficients of α'/ϵ on the two sides of (9) precisely Eqs. (8). In the next (two loop) order the $o((\alpha')^2/\epsilon)$ terms in Eq. (9) contain both the two loop contributions and the new terms originating from M_{ij}; the right hand side of Eqs. (9) is given as:

$$\begin{aligned}\hat{Y}_{ij}(\tilde{g},\tilde{b}) + &\frac{\delta \hat{R}_{ij}}{\delta g_{kl}}|_{\tilde{g},\tilde{b}} M_{(kl)}(\tilde{g},\tilde{b}) + \frac{\delta \hat{R}_{ij}}{\delta b_{kl}}|_{\tilde{g},\tilde{b}} M_{[kl]}(\tilde{g},\tilde{b}) \\ &- \frac{\delta M_{ij}}{\delta g_{kl}}|_{\tilde{g},\tilde{b}} \hat{R}_{(kl)}(\tilde{g},\tilde{b}) - \frac{\delta M_{ij}}{\delta b_{kl}}|_{\tilde{g},\tilde{b}} \hat{R}_{[kl]}(\tilde{g},\tilde{b}) .\end{aligned} \tag{15}$$

Equating these with the $o((\alpha')^2/\epsilon)$ terms on the left hand side of (9) – which can be obtained from the expressions on the r.h.s. of Eq. (8) by replacing the various components of $\hat{R}_{ij}(g,b)$ by the corresponding components of $\hat{Y}_{ij}(g,b)$ – yields an equation for $M_{ij}(g,b)$. We leave the analysis of the resulting equations (the problem of existence of a solution for a general background) for future work as it is somewhat complicated. We shall content ourselves to present below the solution just for the special case of a block diagonal metric which is fairly simple from a calculational point of view, but shows that our modified duality equations (9) admit a non trivial solution.

We would like to emphasize once again that the existence of a non-trivial M_{ij} (necessary for duality be a true quantum symmetry), implies that the dual model cannot be interpreted as a standard σ-model *beyond one loop*, hence the modified duality transformation *does not map* σ-models into σ-models.

Let us now present an explicit construction of the mapping γ for a special class of σ-models, which are the 'block diagonal' purely metric σ-models. In these models b_{ij} and consequently the antisymmetric part of $T^{(0)}_{ij}$ vanishes identically and only the g_{00} and the $g_{\gamma\delta}$ components of the metric (and of $T^{(0)}_{ij}$) are

[1] Since in $T^{(0)}_{ij}$ the residues of the higher order poles in ϵ are determined by that of the single pole, it is enough if Eq. (9) holds order by order in α' for the residues of single poles on the two sides.

different from zero in the adapted coordinate system: $g_{00} = g_{00}(\xi^\gamma)$, $g_{0\gamma} \equiv 0$ and $g_{\gamma\delta} = g_{\gamma\delta}(\xi^\beta)$. For these models the equations following from the modified duality transformation rules Eqs. (9) for M_{ij} simplify considerably:

$$
\begin{aligned}
-\frac{\hat{Y}_{00}(g)}{g_{00}^2} &= \hat{Y}_{00}(\tilde{g}) + \frac{\delta R_{00}}{\delta g_{kl}}|_{\tilde{g}} M_{kl}(\tilde{g}) - \frac{\delta M_{00}}{\delta g_{kl}}|_{\tilde{g}} R_{kl}(\tilde{g}) + \eta^\alpha \partial_\alpha \tilde{g}_{00}\,, \\
\hat{Y}_{\alpha\beta}(g) &= \hat{Y}_{\alpha\beta}(\tilde{g}) + \frac{\delta R_{\alpha\beta}}{\delta g_{kl}}|_{\tilde{g}} M_{kl}(\tilde{g}) - \frac{\delta M_{\alpha\beta}}{\delta g_{kl}}|_{\tilde{g}} R_{kl}(\tilde{g}) + D_\alpha \eta_\beta + D_\beta \eta_\alpha\,,
\end{aligned}
\tag{16}
$$

where R_{ij} denote the ordinary Ricci tensor and η^α describe the above mentioned reparametrization (diffeomorphism) freedom. Eqs. (16) admit the following simple solution:

$$
M_{00}(g) = \frac{g_{00}}{2}(\partial_\alpha \ln(g_{00}))^2\,, \qquad M_{0\beta} = 0\,, \qquad M_{\beta\alpha} = 0\,, \tag{17}
$$

where $(\partial_\beta \ln g_{00})^2$ stands for $g^{\beta\alpha}\partial_\beta \ln g_{00} \partial_\alpha \ln g_{00}$ and $\eta_\alpha = \partial_\alpha(\partial_\beta \ln g_{00})^2/8$. This clearly shows that our proposed modication of the Abelian T-duality transformations (9) is nontrivial. Eqs. (17) coincides with the two loop modification found in Ref. [10].

Knowing an explicit solution of Eq. (9) it is not difficult to see that it does not corresponds to a simple change of the renormalization scheme in the sense of Ref. [15]. In fact there is no choice of the constants k_1 and k_2 in

$$
\hat{g}_{ij} = g_{ij} + \alpha'(k_1 R_{ij} + k_2 g_{ij} R), \tag{18}
$$

describing the most general change in the renormalization scheme compatible with *full target space covariance* that would reduce to Eqs. (17) (even up to reparametrizations) in our block diagonal special case.

3 Examples

The quantum equivalence of dually related models can be studied on the simple but not completely trivial example of the $O(3)$ σ-model described in terms of polar coordinates:

$$
\mathcal{L} = \frac{1}{2\lambda}\left((\partial_\mu\theta)^2 + \sin^2\theta(\partial_\mu\phi)^2\right) = \frac{1}{\lambda}\tilde{\mathcal{L}}, \tag{19}
$$

and its abelian dual (based on the ϕ translation isometry):

$$
\mathcal{L}^d = \frac{1}{2\lambda}\left((\partial_\mu\theta)^2 + \frac{(\partial_\mu\Phi)^2}{\sin^2(\theta)}\right) = \frac{1}{\lambda}\tilde{\mathcal{L}}^d\,. \tag{20}
$$

This model is already sufficient to illustrate some of the main points of this paper. Eq. (19) describes an asymptotitcally free model and we demonstrate below that one indeed has to use the modifications of the duality transformations, Eqs. (9,16,17), to obtain the same β-function from the dual model as from the original one.

We now carry out explicitly the renormalization up to two loops of both the original (19) and the dual theory (20) to see if λ really gets renormalized in the same way. Our general strategy to carry out the renormalization of this type of σ-models and to obtain the corresponding β functions is described quite in some detail in Ref. [4], here we just quote the essential formulae. The procedure is based on the one resp. two loop counterterms for the general σ-models Eqs. (4,5). The loop expansion parameter, α', expressed in terms of the coupling, λ, is $\alpha' = \lambda/(2\pi)$. The explicit form of the one and two loop counterterms can then be written as:

$$\Sigma_1 = \frac{1}{4}\hat{R}_{ij}\Xi^{ij}, \qquad \Sigma_2 = \frac{1}{8}\hat{Y}_{ij}\Xi^{ij}. \tag{21}$$

We convert the previous counterterms into coupling renormalization by assuming that in the one ($i = 1$) and two ($i = 2$) loop orders their bare and renormalized values are related as

$$\lambda_0 = \mu^\epsilon \lambda \Big(1 + \frac{\zeta_1 \lambda}{\pi\epsilon} + \frac{\zeta_2 \lambda^2}{8\pi^2\epsilon} + ... \Big) = \mu^\epsilon \lambda Z_\lambda(\lambda) \,, \tag{22}$$

where the dots stand for both the higher loop contributions and for the higher order pole terms. The unknown ζ_i ($i = 1{,}2$) are determined from the following equations:

$$-\,\zeta_i \tilde{\mathcal{L}} + \frac{\delta \tilde{\mathcal{L}}}{\delta \xi^k} \xi_i^k(\xi) = \Sigma_i \,. \tag{23}$$

As discussed in Ref. [4] Eqs. (23) admits a simple interpretation: the general counterterms may be absorbed by the renormalization of the coupling together with a (in general non-linear) redefinition of the fields ξ^j:

$$\xi_0^j = \xi^j + \frac{\xi_1^j(\xi^k)\lambda}{\pi\epsilon} + \frac{\xi_2^j(\xi^k)\lambda^2}{8\pi^2\epsilon} + ... \,, \tag{24}$$

where ξ_1^j, ξ_2^j have to satisfy Eqs. (23). In the special case when ξ_i^k depends linearly on ξ i.e. $\xi_i^k(\xi) = \xi^k y_i^k$, Eqs. (24) simplify to an ordinary multiplicative wave function renormalization. We emphasize that it is not a priori guaranteed that Eqs. (23) may be solved at all for ζ_i and the functions $\xi_i^k(\xi)$. If Eqs. (23) do not have a solution, then the renormalization of the model is not possible within the restricted subspace characterized by the coupling λ in the (infinite dimensional) space of metrics. On the other hand, if Eqs. (23) admit a solution, then, writing $Z_\lambda = 1 + y_\lambda(\lambda)/\epsilon + ...$ the β function of λ is readily obtained:

$$\mu \frac{d\lambda}{d\mu} = \beta_\lambda = \lambda^2 \frac{\partial y_\lambda}{\partial \lambda} \,. \tag{25}$$

For the $O(3)$ σ-model the explicit form of the counterterms

$$\Sigma_1 = \frac{1}{4}\Big((\partial_\mu\theta)^2 + \sin^2\theta (\partial_\mu\phi)^2 \Big) \,, \qquad \Sigma_2 = 2\Sigma_1, \tag{26}$$

implies, that the θ and ϕ fields undergo no renormalization. Eqs. (23) give in this case $\zeta_1 = -\frac{1}{2}$, $\zeta_2 = -1$, and using them in Eq. (25) leads to the well known β-function : $\beta_\lambda = -\frac{\lambda^2}{2\pi}(1 + \frac{\lambda}{2\pi})$.

For the dual model, Eq. (20), the counterterms have a slightly more complicated form:

$$\begin{aligned}\Sigma_1 &= -\frac{1}{4}\frac{1+\cos^2\theta}{\sin^2\theta}\left((\partial_\mu\theta)^2 + \frac{(\partial_\mu\Phi)^2}{\sin^2\theta}\right),\\ \Sigma_2 &= \frac{1}{2}\frac{(1+\cos^2\theta)^2}{\sin^4\theta}\left((\partial_\mu\theta)^2 + \frac{(\partial_\mu\Phi)^2}{\sin^2\theta}\right),\end{aligned} \tag{27}$$

Looking at these counterterms we see that in principle in the present renormalization problem we can have an ordinary wavefunction renormalization for Φ and a redefinition of the variable θ, i.e. we have:

$$\begin{aligned}\lambda_0 =& \mu^\epsilon\lambda\Big(1 + \frac{\zeta_1\lambda}{\pi\epsilon} + \frac{\zeta_2\lambda^2}{8\pi^2\epsilon} + ..\Big),\\ \theta_0 =& \theta + \frac{T_1(\theta)\lambda}{\pi\epsilon} + \frac{T_2(\theta)\lambda^2}{8\pi^2\epsilon} + ...\\ \Phi_0 =& \Phi\Big(1 + \frac{z_1\lambda}{\pi\epsilon} + \frac{z_2\lambda^2}{8\pi^2\epsilon} + ..\Big).\end{aligned} \tag{28}$$

At one loop Eqs. (23) yields now the following equations:

$$2T_1' - \zeta_1 = -\frac{1+\cos^2\theta}{2\sin^2\theta}, \qquad \frac{2z_1 - \zeta_1}{\sin^2\theta} - \frac{2T_1\cos\theta}{\sin^3\theta} = -\frac{1+\cos^2\theta}{2\sin^4\theta}, \tag{29}$$

while at two loops one obtains:

$$2T_2' - \zeta_2 = \frac{(1+\cos^2\theta)^2}{\sin^4\theta}, \qquad \frac{2z_2 - \zeta_2}{\sin^2\theta} - \frac{2T_2\cos\theta}{\sin^3\theta} = \frac{(1+\cos^2\theta)^2}{\sin^6\theta}. \tag{30}$$

The two equations appearing in (29) have a consistent solution

$$T_1(\theta) = \mathrm{cotg}\theta/2, \qquad \zeta_1 = -1/2, \qquad z_1 = -1/2, \tag{31}$$

which shows that at one loop the β functions of (19) and of (20) are indeed the same, while at the two loop order we meet the problem exhibited already in several examples in Ref. [4], that is there is no choice of ζ_2 and z_2 that would guarantee that T_2 expressed algebraically from the second equation in (30) would also solve the differential equation in (30). Thus renormalizing the dual model described by Eq. (20) as a standard σ-model one finds that it is not renormalizable, therefore it cannot be equivalent to the $O(3)$ model given by Eq. (19).

The $O(3)$ σ-model belongs to the class of block diagonal purely metric σ-models therefore we now apply the modified duality transformation Eqs. (17) to demonstrate explicitly how the two loop 'anomaly' is removed in our framework. In fact taking into account the explicit modification of Eqs. (2) following from Eqs. (9, 17) changes the two loop equations, Eqs. (30), as:

$$2T_2' - \zeta_2 = 1 - \frac{4(1+2\cos^2\theta)}{\sin^4\theta}, \quad \frac{2z_2 - \zeta_2}{\sin^2\theta} - \frac{2T_2\cos\theta}{\sin^3\theta} = -\frac{\sin^4\theta + 4\cos^2\theta}{\sin^6\theta}. \tag{32}$$

Remarkably this system admits a consistent solution:

$$T_2(\theta) = 2\frac{\cos\theta}{\sin^3\theta}\,, \qquad \zeta_2 = -1\,, \qquad z_2 = -1, \tag{33}$$

showing that in the new framework the dual of the $O(3)$ σ-model leads to the same β-function as the original model even at the two loop level, and that the modifications of the 'naive' Abelian duality transformation rules, Eqs. (2), are essential, indeed.

Finally we show that as far as the coupling constant renormalization is concerned, the non trivial γ mapping, Eq. (17), is also necessary to establish the two loop equivalence between

$$\mathcal{L} = \frac{1}{2\lambda}\left((\partial_\mu r)^2 + r^2(\partial_\mu\phi)^2\right) \tag{34}$$

describing two free scalar fields in polar coordinates, and its abelian dual:

$$\mathcal{L}^d = \frac{1}{2\lambda}\left((\partial_\mu r)^2 + r^{-2}(\partial_\mu\Phi)^2\right) = \frac{1}{\lambda}\tilde{\mathcal{L}}^d\,. \tag{35}$$

Eq. (34) describes a (not very complicated) CFT and some evidence was given in Ref. [6], using the "minisuperspace" approximation, that the dual model is indeed equivalent to the original one (see also Ref. [16]). In our framework the equivalence hinges upon whether the coupling constant of the dual theory really gets renormalized as indicated by the non trivial metric or stays unrenormalized as in the original free model. In applying the coupling constant renormalization we note that Eq. (35) is invariant under the $\lambda\to a\lambda$, $r\to a^{1/2}r$, $\Phi\to a\Phi$ scaling transformations, thus we can effectively set the wavefunction renormalization of Φ to one, i.e. we have:

$$\begin{aligned}\lambda_0 =& \mu^\epsilon\lambda\Big(1+\frac{\zeta_1\lambda}{\pi\epsilon}+\frac{\zeta_2\lambda^2}{8\pi^2\epsilon}+..\Big),\\ r_0 =& r+\frac{r_1(r)\lambda}{\pi\epsilon}+\frac{r_2(r)\lambda^2}{8\pi^2\epsilon}+...\end{aligned} \tag{36}$$

At one loop we get from Eqs. (23) a system that admits the solution $r_1(r) = 1/(2r)$, $\zeta_1 = 0$, however at two loops it yields

$$2r_2' - \zeta_2 = \frac{4}{r^4}, \qquad -\frac{\zeta_2}{r^2} - \frac{2r_2(r)}{r^3} = \frac{4}{r^6}\,, \tag{37}$$

which admits *no* solution. Taking into account the terms coming from the non trivial part of the γ mapping changes Eqs. (37) to

$$2r_2' - \zeta_2 = \frac{9}{r^4}\,, \qquad -\frac{\zeta_2}{r^2} - \frac{2r_2(r)}{r^3} = \frac{3}{r^6}\,, \tag{38}$$

which admits a consistent solution: $r_2(r) = -3/(2r^3)$, $\zeta_2 = 0$; showing that the dual of the free model remains 'free' even at the two loop level.

Based on the above (admittedly as yet incomplete) evidence, that our proposed modified duality transformations, (9), do restore the equivalence between

dual σ-models in perturbation theory, we expect that for the general case (i.e. a not necessarily block diagonal metric tensor $g_{0\alpha} \neq 0$ and $b_{ij} \neq 0$) Eqs. (9) also admit a solution. Then it is natural to conjecture, that the modified duality transformations restore the equivalence between the original and the dual models at two loops for the example discussed in Sect. 4.2 of Ref. [4] just as in the $O(3)$ case. Furthermore we also expect that a similar modification of the non-Abelian duality transformations restore the two-loop equivalence between the principal σ-model and its nonabelian dual (for the two-loop problem in that case, see Sect. 5 of Ref. [4]). (Recently the quantum equivalence between gauged WZW models and their non-abelian duals has been proved in Ref. [17].) Finally we think it would be interesting to investigate the γ mapping beyond two loops.

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